COLOR-AVOIDING RAMSEY FOR DIRECTED PATHS AND VECTOR SEQUENCES

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ABSTRACT. We study two closely related Ramsey-type questions. The first is about vectors: how long can a sequence of vectors in $[n]^q$ be if, for all pairs in the sequence, the later vector is strictly larger than the earlier in at least $q-1$ coordinates? The second is graph-theoretic: given a q-edge-coloring of a tournament, how long of a directed path can we guarantee whose edges avoid one of the colors? These problems arise naturally in many areas, such as convex geometry and extremal hypergraph theory, and have been extensively studied over the past 50 years.

We prove that if $\varepsilon > 0$ is fixed and q is sufficiently large, then every N-vertex tournament contains a color-avoiding directed path of length $N^{1-\varepsilon}$. Consequently, the maximum length of a $(q-1)$-increasing sequence of vectors in $[n]^q$ has length at most $n^{1+\varepsilon}$. Our results answer a question of Gowers and Long, strengthen several of their results, and extend earlier works of Tiskin and Loh.

1. INTRODUCTION

One of the earliest, and most influential, results in Ramsey theory is the Erdős–Szekeres lemma [15], which states that every sequence of N real numbers contains a monotone subsequence of length at least $\sqrt{N}$. This innocent statement turns out to have deep connections to discrete geometry [15], high-dimensional partitions [27], Ramsey theory of paths [19], and sequences of vectors [31]; the latter two will be of particular importance to us, as we shortly explain.

The Erdős–Szekeres lemma has an equivalent formulation in terms of edge colorings: it states that in every 2-edge-coloring of the vertices of a complete graph with vertex set $[N]$, there is a monochromatic *monotone* path of length $\sqrt{N}$; here, and throughout the paper, a path is *monotone* if its vertices come in increasing order, and the *length* of a path refers to its number of vertices. A particularly elegant proof of this statement, which explains the connections to relations on vectors, was given by Seidenberg [31]: one simply records for each vertex the length of the longest monotone monochromatic path ending at that vertex in each color, and notes that these vectors must be pairwise distinct, from which the result follows by the pigeonhole principle. More precisely, given two vectors $x, y \in \mathbb{R}^q$, let us write that $x <_1 y$ if there is at least one coordinate $i \in [q]$ such that $x_i < y_i$; then Seidenberg's argument shows that the Erdős–Szekeres lemma is equivalent to the statement that if a sequence of vectors $x_1, \dots, x_N \in [n]^2$ has the property that $x_a <_1 x_b$ for all $a < b$, then $N \leqslant n^2$. And this last statement is essentially trivial, since the pigeonhole principle implies that if $N > n^2$, then two vectors in the sequence must be equal, which clearly violates this 1-increasing property.

There are numerous ways of generalizing these results. For example, the q-dimensional analogue of the argument above immediately implies that if we color $E(K_N)$ with q colors, then we must always have a monochromatic monotone path of length at least $N^{1/q}$. For another natural extension, we can look at arbitrary tournaments; note that a monotone path in K_N is precisely the same as

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a directed path in the *transitive* N-vertex tournament, but the same question makes sense in any tournament. In particular, as observed by Chvátal and Gyárfás–Lehel [9, 23] it is not hard to use the Gallai–Hasse–Roy–Vitaver theorem [21, 25, 29, 38] to show that no matter how we q-color the edges of an N-vertex tournament, there is a monochromatic directed path of length at least $N^{1/q}$. In this setting, the corresponding vector question asks about the maximum size of a set $\{x_1, \ldots, x_N\} \subseteq [n]^q$ with the property that for all $a \neq b$, either $x_a <_1 x_b$ or $x_b <_1 x_a$; that is, we still want every pair to be strictly comparable in at least one coordinate, but we do not insist that the sequence be increasing according to this notion. This naturally corresponds to the structure of an arbitrary tournament, recording for each pair (a, b) whether $x_a <_1 x_b$ or $x_b <_1 x_a$.

In this paper, we study a close cousin of these classical questions, which turns out to be surprisingly rich and difficult. On the level of vectors, we study a generalization of the relation $<_1$ considered above, in which we ask whether one vector is strictly greater than another on at least r of the coordinates. On the level of graph theory, we move from the classical Ramsey-theoretic goal of finding a monochromatic structure, to the alternative goal of finding a *color-avoiding* structure; that is, one which receives at most $q - 1$ colors out of the palette of q colors used in total. We now turn to discuss these problems, and our results, in more detail.

1.1. **Sequences of vectors.** Let $q \geqslant r \geqslant 1$ be integers. Extending the definition above, we define the relation $<_r$ on $\mathbb{R}^q$ by saying that $x <_r y$ if there are at least r coordinates $i \in [q]$ for which $x_i < y_i$. Despite the notation, $<_r$ is not a partial order whenever $r < q$, as it is not transitive. There are (at least) two natural extremal questions one can ask about this relation, both of which have a long and rich history. First, following the notation of Gowers and Long [22], we define $F_{q,r}(n)$ to be the maximum number of vectors $x_1, \ldots, x_N \in [n]^q$ such that for all $a < b$, we have $x_a <_r x_b$. In other words, $F_{q,r}(n)$ measures the maximum length of an *r-increasing* sequence of vectors in $[n]^q$. Second, we also study a weaker notion, of *r-comparable* sets of vectors; formally, we define $G_{q,r}(n)$ to be the maximum size of a set $\{x_1, \ldots, x_N\} \subseteq [n]^q$ such that for all $a \neq b$, either $x_a <_r x_b$ or $x_b <_r x_a$. Given such an r-comparable set, we naturally obtain an N-vertex tournament[1], recording for each pair (a, b) whether $x_a <_r x_b$ or $x_b <_r x_a$. Thus, the r-comparable set is in fact an r-increasing sequence if and only if this tournament is transitive. Additionally, the fact that the r-comparable condition is weaker immediately implies that $F_{q,r}(n) \leqslant G_{q,r}(n)$ for all q, r, n.

The first non-trivial case of this problem is $(q, r) = (3, 2)$, which has been extensively studied for over 50 years, with different researchers arriving at this question from a number of different perspectives, including extremal problems in convex geometry [6] and hypergraphs [22], k-majority tournaments [26], incidence geometry [36], the Erdős–Hajnal conjecture [39], and packing questions motivated by group theory [33] and error-correcting codes [24]. For a detailed discussion of these disparate instantiations of this problem, we refer to [3, Section 4.3]. For this problem, it is quite easy to observe the bounds

$$(1 - o(1))n^{3/2} \leqslant F_{3,2}(n) \leqslant G_{3,2}(n) \leqslant n^2. \tag{1}$$

The lower bound follows from an iterative product construction, which shows that $F_{3,2}(n) \geqslant n^{3/2}$ whenever n is a perfect square (see e.g. [22, Proposition 1.1] for details). The upper bound follows from the simple observation that if we delete the last coordinate from a 2-comparable set in $[n]^3$, we obtain a 1-comparable set in $[n]^2$, which in particular must consist of distinct vectors. The same argument shows that for all integers $q \geqslant r > t$, we have

$$F_{q,r}(n) \leqslant F_{q-t,r-t}(n) \qquad \text{and} \qquad G_{q,r}(n) \leqslant G_{q-t,r-t}(n), \tag{2}$$

[1]Strictly speaking, one only obtains a well-defined tournament if $r > q/2$, as this condition guarantees that if $x <_r y$, then $y \not<_r x$. If $r \leqslant q/2$, then we can, for example, allow antiparallel edges in case $x <_r y$ and $y <_r x$.

since by deleting the last t coordinates of an r-increasing sequence (resp. an r-comparable set) in $[n]^q$, we obtain an $(r-t)$-increasing sequence (resp. an $(r-t)$-comparable set) in $[n]^{q-t}$.

Improving the lower bound on $F_{3,2}(n)$ in (1) appears to be extremely difficult, and Gowers–Long [22, Conjecture 1.2] conjecture that in fact, $F_{3,2}(n) \leqslant n^{3/2}$ for all n. However, quite surprisingly, weakening the problem from 2-increasing sequences in $[n]^3$ to 2-comparable subsets of $[n]^3$ makes a big difference, and already in 1984 Hamaker and Stein [24] found a 2-comparable subset of $[n]^3$ of size $n^{1.51}$, implying that $G_{3,2}(n) \geqslant n^{1.51}$ for sufficiently large n. The exponent in the lower bound was steadily improved (e.g. [34, 35, 37]), always by finding a construction for some fixed n, and then using the same product argument to boost this construction to all large n. Unaware of these prior developments, Gowers and Long [22] developed a continuous relaxation of this problem, allowing them to improve the lower bound to $G_{3,2}(n) \geqslant n^{1.54}$; they even speculate, based on some computational evidence of Wagner, that their construction may be optimal.

On the upper bound side, the main question left open in [24, 34] is whether $G_{3,2}(n) = o(n^2)$. This was finally proved by Tiskin [37], via a rather simple argument: if one views the elements of a 2-comparable set of vectors in $[n]^3$ as the edges of a tripartite 3-uniform hypergraph on vertex set $[n] \sqcup [n] \sqcup [n]$, then it is not hard to check that this hypergraph does not 6 vertices spanning 3 edges. By applying the famous $(6,3)$ theorem of Ruzsa and Szemerédi [30], one concludes that $G_{3,2}(n) = o(n^2)$. However, the savings over the trivial upper bound are very minor; even by applying the strongest known quantitative version of the $(6,3)$ theorem, due to the first author [17], one only obtains that $G_{3,2}(n) \leqslant n^2/2^{\Omega(\log^* n)}$, where $\log^* n$ is the iterated logarithm function. To date, this remains the best known upper bound on $G_{3,2}(n)$.

The only other improvement on the trivial bounds in (1) to date is due to Gowers and Long, who proved that

$$F_{3,2}(n) \leqslant n^{2-\delta}, \tag{3}$$

for an extremely small, but absolute, constant $\delta > 0$. Their proof is based on an induction argument, and the base case of the induction is established by applying the bound $F_{3,2}(n) \leqslant G_{3,2}(n) \leqslant n^2/2^{\Omega(\log^* n)}$ discussed above; since this improvement only beats the trivial bound for an enormous n, the constant δ they obtain is unbelievably small.

For larger values of q, r, the simple bounds corresponding to (1) are

$$(1-o(1))n^{q/r} \leqslant F_{q,r}(n) \leqslant G_{q,r}(n) \leqslant n^{q-r+1}.$$

The lower bound again follows from the same product construction, and the upper bound follows by applying (2) with $t = r-1$, as well as the trivial bound $G_{q-r+1,1}(n) \leqslant n^{q-r+1}$. In general, the lower bound is far from tight; Gowers and Long [22, Lemma 5.1] noted that if q is large and $r < (\frac{1}{2} - \gamma)q$ for some fixed $\gamma > 0$, then a simple probabilistic argument gives a much better lower bound than $n^{q/r}$ for n fixed; one can then use the same product construction to boost this lower bound to larger values of n. However, for $r > q/2$, as far as we are aware, the best known lower bound remains $F_{q,r}(n) \geqslant n^{q/r}$. For the special case $r = q-1$, these bounds become

$$(1-o(1))n^{q/(q-1)} \leqslant F_{q,q-1}(n) \leqslant G_{q,q-1}(n) \leqslant n^2, \tag{4}$$

and thus the gap in the exponent widens as q grows. For upper bounds, one can apply the simple reductions (2) to obtain a power savings when $r = q-1$, as $F_{q,q-1}(n) \leqslant F_{3,2}(n) \leqslant n^{2-\delta}$, by the result of Gowers and Long [22]. However, no such power savings was known for the more general r-comparable question; the only upper bound, which follows from the hypergraph removal lemma as in Tiskin's argument, is $G_{q,r}(n) = o(n^{q-r+1})$, where the savings is extraordinarily small (the savings is roughly the reciprocal of the $(q-r+1)$th level of the Ackermann hierarchy, evaluated at n). As such, Gowers and Long highlighted the following question, saying it is the "most annoying" question they were unable to answer.

Question 1.1 ([22, Question 6.3])**.** *Is there any pair (q,r) for which one can prove a power savings $G_{q,r}(n) \leqslant n^{q-r+1-\delta}$, for some absolute constant $\delta > 0$?*

Our first result answers this question in a very strong form: when $r = q-1$ and q is large, we show that the behavior of both $F_{q,q-1}(n)$ and $G_{q,q-1}(n)$ are close to the lower bound in (4): the true exponent tends to 1 as q grows.

Theorem 1.2. *For $q \geqslant 4$ and any n, we have*

$$F_{q,q-1}(n) \leqslant G_{q,q-1}(n) \leqslant n^{1+O(1/\sqrt{\log q})}.$$

In particular, for any fixed $\varepsilon > 0$ and any sufficiently large $q \geqslant q_0(\varepsilon)$, we have

$$F_{q,q-1}(n) \leqslant G_{q,q-1}(n) \leqslant n^{1+\varepsilon}.$$

Theorem 1.2 is our main result on this vector problem, asymptotically resolving, as $q \to \infty$, the growth rate of $F_{q,q-1}(n)$ and $G_{q,q-1}(n)$. However, if one is only interested in answering Question 1.1, there is a much simpler, and quite surprising, argument, which proves the weaker bound

$$G_{q,q-1}(n) \leqslant n^{2-\delta} \text{ for all } q \geqslant 4, \text{ where } \delta > 0 \text{ is an absolute constant.} \tag{5}$$

In fact, (5) follows from the following proposition, which shows that when $r > \frac{2q}{3}$, there is no difference between the r-increasing and r-comparable questions.

Proposition 1.3. *Let $q \geqslant r$ be integers with $r > \frac{2q}{3}$. Then $F_{q,r}(n) = G_{q,r}(n)$ for all n.*

Note that Proposition 1.3 immediately implies (5): for $q \geqslant 4$, we have the chain of inequalities

$$G_{q,q-1}(n) = F_{q,q-1}(n) \leqslant F_{3,2}(n) \leqslant n^{2-\delta},$$

from Proposition 1.3, (2), and (3), respectively. That is, Proposition 1.3 allows us to reduce to the theorem of Gowers and Long, and gives us an extremely small, but positive, power saving on the trivial upper bound on $G_{q,q-1}(n)$.

Proposition 1.3 is quite surprising, since, as discussed above, it is probably false for $(q,r) = (3,2)$; we have that $G_{3,2}(n) \geqslant n^{1.54}$, whereas Gowers and Long conjectured $F_{3,2}(n) \leqslant n^{3/2}$. This also suggests that the requirement $r > \frac{2q}{3}$ in Proposition 1.3 is best possible. Despite this, the proof of Proposition 1.3 is extremely simple.

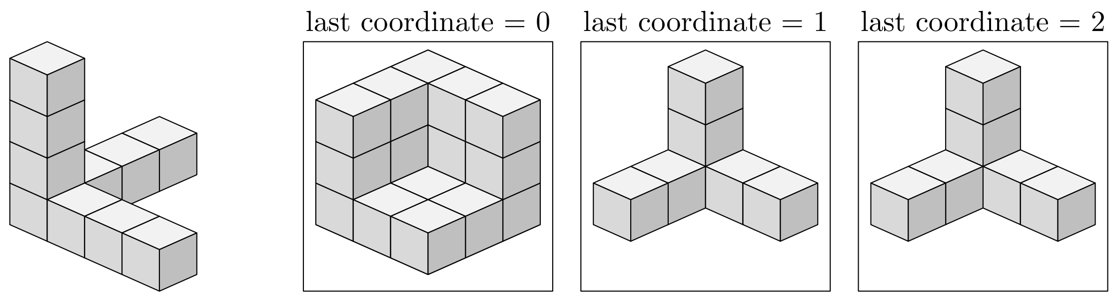
Proof of Proposition 1.3. Let $S \subseteq [n]^q$ be an r-comparable set with $|S| = G_{q,r}(n)$. If the tournament defined by S is transitive, then $|S| \leqslant F_{q,r}(n)$, and we are done. If not, then this tournament has a cyclic triangle, that is, three vectors $x,y,z \in [n]^q$ such that $x <_r y <_r z <_r x$. Let

$$A = \{i \in [q] : x_i < y_i\}, \qquad B = \{i \in [q] : y_i < z_i\}, \qquad C = \{i \in [q] : z_i < x_i\}.$$

Then $|A|,|B|,|C| \geqslant r > \frac{2q}{3}$, hence $A \cap B \cap C \neq \varnothing$. But if $i \in A \cap B \cap C$, we have $x_i < y_i < z_i < x_i$, which is a contradiction. $\square$

1.2. **Packing pods in q-dimensional space.** A *tripod* of order n consists of a corner and the three adjacent edges of an integer $n \times n \times n$ cube. The corner of the tripod is called its *apex*. We say that a collection of tripods is *aligned* if they are translates of one other, and that they *pack* if they are pairwise disjoint. The maximum number of aligned tripods which pack and whose apex is in $[n]^3$ is precisely $G_{3,2}(n)$, as two tripods with apices in $[n]^3$ are disjoint if and only if their apices are 2-comparable. Recall that Tiskin [37] proved that $G_{3,2}(n) = o(n^2)$; his motivation came from the question of packing tripods, and he proved the equivalent statement that the density of 3-space that can be filled with tripods of order of n is $o(1)$ as $n \to \infty$. Indeed, it is not hard to show that the two questions are equivalent: bounding the maximum number of disjoint tripods of order n whose apex lies in $[n]^3$ is equivalent to bounding the maximum density of 3-space that can be packed by order-n tripods. For a formal statement (and in particular for the formal definition of the density of a packing), we refer to [35, Chapter 4] or [32].

More generally, a (q,r)-*pod of order* n consists of a corner and the incident $(r-1)$-dimensional faces of an integer q-dimensional cube with side length n. We may just refer to it as a (q,r)-pod if the order n is implicit. As above, the corner of the (q,r)-pod is called its *apex*, and a collection of (q,r)-pods is *aligned* if they are translates of each other, and the collection *packs* if they are pairwise disjoint. The maximum number of aligned (q,r)-pods that pack whose apex is in $[n]^q$ is precisely $G_{q,r}(n)$ as two (q,r)-pods with apices in $[n]^q$ are disjoint if and only if their apices are r-comparable. Indeed, if $(x_1,\dots,x_q),(y_1,\dots,y_q)\in[n]^q$ are not r-comparable, then $(z_1,\dots,z_q)\in[n]^q$ given by $z_i=\max(x_i,y_i)$ for each coordinate i is in both the (q,r)-pod with apex $(x_1,\dots,x_q)$ and the (q,r)-pod with apex $(y_1,\dots,y_q)$. Conversely, if $(x_1,\dots,x_q),(y_1,\dots,y_q)$ are r-comparable, then without loss of generality, $y_i>x_i$ for $i=1,\dots,r$, in which case every element of the (q,r)-pod with apex $(y_1,\dots,y_q)$ has value at least y_i in coordinate i for each $i\in[r]$, but no element of the (q,r)-pod with apex $(x_1,\dots,x_q)$ has this property, so these (q,r)-pods are disjoint.



(A) A tripod of order 4 (B) A $(4,3)$-pod of order 3; each square represents a 3-dimensional slice of $\mathbb{R}^4$

As above, it is not hard to convert between the question of packing (q,r)-pods of order n with apices in $[n]^q$ and the question of densely packing q-dimensional space with (q,r)-pods of order n. From Theorem 1.2 we thus get a stronger bound on the density of q-space that can be packed by $(q,q-1)$-pods for larger q.

Corollary 1.4. *For n sufficiently large as a function of q, the density of q-space that can be packed by $(q,q-1)$-pods of order n is at most $n^{O(1/\sqrt{\log q})-1}$.*

Here we used the fact that there are only 2^q corners of the q-cube, so by losing a factor 2^q, we can assume that all the $(q,q-1)$-pods we pack are aligned.

1.3. **Color-avoiding paths.** Our proof of Theorem 1.2 uses the language of color-avoiding paths in tournaments. This is both a much more convenient setting in which to place our arguments, and is also a setting with a large number of natural questions. We now turn to discuss these problems and our new results on them.

The kind of question we study, wherein one is given a colored structure, and wishes to find a substructure avoiding one of the colors, has a rich history in Ramsey theory. The study of such questions goes back at least to the seminal work of Erdős and Szemerédi [16], and is a special case of the "generalized Ramsey theory" introduced by Erdős and Gyárfás [14] and intensively studied since. We also refer to the recent paper [28], which studies a color-avoiding question in hypergraphs that is closely related to our main topic.

The most basic question of the kind we study was introduced by Loh [26], who asked for the length of the longest color-avoiding monotone path that is guaranteed in every 3-coloring of $E(K_N)$. The trivial lower bound here is $\sqrt{N}$, obtained by simply merging two of the colors and applying the Erdős–Szekeres lemma to find a monochromatic monotone path in the modified coloring; any such path must receive at most two colors in the original coloring, and hence is color-avoiding. On the other hand, the best known upper bound for this problem is $N^{2/3}$, which is obtained by a simple product construction (see Proposition 3.3 for details).

The similarity of these bounds to (1) is not a coincidence. Indeed, Loh observed that one can convert this into a vector problem, by assigning to every vertex a of K_N a vector $x_a \in \mathbb{N}^3$, where the ith coordinate of x_a is the length of the longest monotone path ending at a which avoids the ith color. The key observation is that if $a < b$ and the edge ab receives color i, then for both choices of $j \in [3] \setminus \{i\}$, we can extend a longest j-avoiding path ending at a to a strictly longer one ending at b. This implies that $(x_a)_j < (x_b)_j$ for both choices of j, and hence $x_a <_2 x_b$. In particular, if the longest color-avoiding monotone path has length at most n, then we have found a 2-increasing sequence of vectors in $[n]^3$, implying that $N \leqslant F_{2,3}(n)$. As it turns out (see Proposition 3.1), this connection can be reversed, and the study of $F_{3,2}(n)$ is equivalent to the study of the longest color-avoiding monotone path in a 3-coloring of $E(K_N)$. More generally, $F_{q,q-1}(n)$ corresponds to finding color-avoiding monotone paths in q-colorings of $E(K_N)$.

As a consequence, one can convert the results discussed previously to this setting. Tiskin's [37] argument (observed independently by Loh [26] in this context) allows us to improve the trivial bound above, and always find a color-avoiding monotone path of length $\omega(\sqrt{N})$. More strongly, the bound (3) of Gowers and Long [22] implies, for $q > 3$, that every q-edge-coloring of $E(K_N)$ contains a color-avoiding path of length $N^{\frac{1}{2}+\delta}$, for some absolute constant $\delta > 0$. On the other hand, the best known upper bound for this problem, corresponding to the lower bound on $F_{q,q-1}(n)$, is that there is a q-coloring of $E(K_N)$ whose longest color-avoiding directed path has length $N^{1-1/q}$. The gap between the exponents for large q is equivalent to that in (4). Our main theorem in this setting, which is equivalent to Theorem 1.2, shows that the true exponent tends to 1 as q grows.

Theorem 1.5. *Let $q \geqslant 4$. In any q-coloring of $E(K_N)$, there exists a color-avoiding monotone path of length at least $N^{1-O(1/\sqrt{\log q})}$.*

As we saw in our discussion of the Erdős–Szekeres lemma, statements about monotone paths are equivalent to statements about directed paths in transitive tournaments. More generally, it is thus natural to ask such questions for arbitrary tournaments. Our original motivation to study this more general question came from the study of r-comparable sequences, whose structure naturally corresponds to an arbitrary tournament. Proposition 1.3 implies that in the vector setting, there is nothing new in this seemingly more general problem for $r > \frac{2q}{3}$, but in the Ramsey-theoretic setting, working with general tournaments seems to be genuinely harder and more general.

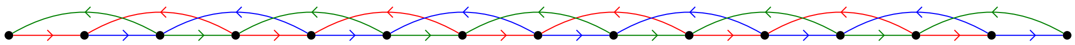
Our second main result is a strengthening of Theorem 1.5, obtaining essentially the same conclusion for all tournaments.

Theorem 1.6. *Let $q \geqslant 4$. In any q-edge coloring of any N-vertex tournament, there exists a color-avoiding directed path of length at least $c_q N^{1-O(1/\sqrt{\log q})}$, where $c_q > 0$ is a constant depending only on q.*

Note that the conclusion of Theorem 1.6 is (very) slightly weaker than that of Theorem 1.5, because of the presence of the constant c_q. When working with transitive tournaments, we can eliminate this constant via a powering trick (very similar to the use of the tensor product trick in the study of Sidorenko's conjecture, as in e.g. [1, 13]), but this does not appear to be possible when working with arbitrary tournaments.

In the setting of vectors, Proposition 1.3 implies that $F_{q,q-1}(n) = G_{q,q-1}(n)$ for all $q \geqslant 4$, i.e. that the hardest case of the $(q-1)$-comparable problem is actually the case when the tournament defined by the vectors is transitive. We do not know how to prove such a result in the setting of color-avoiding paths, nor do we even expect it to be true in general. Nevertheless, we are able to prove an approximate form of Proposition 1.3 in this setting, which says that the problem of color-avoiding directed paths really is much easier if the tournament is *far* from transitive. To make this precise, let us recall that an N-vertex tournament is *δ-close to transitive* if it can be made transitive by reversing the orientation of at most δN^2 edges, and is *δ-far from transitive* otherwise.

Theorem 1.7. *If an N-vertex tournament is δ-far from transitive, then any q-coloring of its edges contains a directed path of length $c\delta^2 N/q^3$ which is colored by at most three colors, where $c > 0$ is an absolute constant. Furthermore, the coloring contains the square of a path such that the direction of the edges of consecutive vertices are forward, the direction of edges between vertices of distance two are backwards, and the coloring of the edges is periodic with period 3.*



In particular, if $q \geqslant 4$ is fixed and $\delta = \Omega(1)$, then we find a color-avoiding directed path of length $\Omega(N)$, much stronger than the result in Theorem 1.6. We will use this in our proof of Theorem 1.6, as it allows us to focus on the nearly transitive case. We stress that there is a genuine difference between the $q = 3$ and $q \geqslant 4$ cases: the construction of Gowers–Long is $\Omega(1)$-far from transitive, yet does not have color-avoiding directed paths of linear length. Moreover, their construction has much shorter color-avoiding paths than the best known, and conjecturally extremal, transitive construction. We also stress that, in general, we do not expect transitive tournaments to be the ones minimizing the length of color-avoiding paths.

The proof of Theorem 1.7 is extremely short and simple, and relies on a result of the first two authors [20] on the structure of tournaments that are far from transitive. They proved a directed version of the celebrated triangle removal lemma [30] with very good quantitative bounds, and it is this good quantitative behavior that means that the δ-dependence in Theorem 1.7 is polynomial.

The rest of this paper is organized as follows. We present a high-level outline of the proof of Theorem 1.6 in Section 2. In Section 3, we warm up to the topic by presenting some simple proofs, including the short proof of Theorem 1.7, proofs that the vector and Ramsey notions are equivalent, and some properties of the product construction alluded to previously; in particular, we also show there how Theorem 1.6 implies Theorems 1.2 and 1.5. Finally, we present the full proof in Section 4.

In this paper, all logarithms are to base 2. We systematically omit floor and ceiling signs whenever they are not crucial.

2. Proof overview

In this section, we give a sketch of the proof of Theorem 1.6. For the majority of our discussion, we focus on the case of *transitive* tournaments; most of the ideas used in the proof of the full theorem are already present in this setting, with fewer technical complications. At the end of the sketch, we describe which additional ingredients are needed for the general case.

As our tournament is transitive, we can simply view it as an ordered complete graph on the vertex set $[N]$; additionally, a path in this complete graph is directed if and only if it is monotone, i.e. its vertices are strictly increasing as integers in $[N]$. So our task is to prove that any q-edge-colored complete graph on $[N]$ contains a color-avoiding monotone path of length at least $N^{1-C/\sqrt{\log q}}$, and we prove this by induction on N. For this proof sketch, we will not worry too much about the precise quantitative error term, and only aim to prove a lower bound of the form $N^{1-o(1)}$, for some function $o(1)$ that tends to 0 as $q \to \infty$. For a q-edge-colored transitive tournament T and a color $i \in [q]$, let $\ell_i(T)$ denote the length of the longest directed path in T avoiding color i. Thus, our goal is to prove by induction that $\ell_i(T) \geqslant N^{1-o(1)}$ for some i. We will actually strengthen the induction hypothesis, and prove that $\Pi(T) \geqslant N^{q-o(q)}$, where we define $\Pi(T) = \prod_{i=1}^{q} \ell_i(T)$. The statement $\Pi(T) \geqslant N^{q-o(q)}$ immediately implies that $\ell_i(T) \geqslant N^{1-o(1)}$ for some $i \in [q]$, hence this really is a stronger statement.

Our strategy is as follows. We divide $[N]$ into two subintervals $T_1 = [1, N/2]$ and $T_2 = (N/2, N]$, and apply the inductive hypothesis in each interval. Our dream scenario is to take a longest i-avoiding path P_1 in the first half and a longest i-avoiding path P_2 in the second half, and to glue them together to obtain an i-avoiding path of double the length. Of course, for this to work, we need the edge joining the end of P_1 and the start of P_2 to *not* be colored with color i; if we can ensure this, then their concatenation $P_1 \cup P_2$ truly is an i-avoiding path of length $|P_1|+|P_2|$. Thus, if we can do this for color i, we learn that

$$\ell_i(T) \geqslant \ell_i(T_1) + \ell_i(T_2) \geqslant 2\sqrt{\ell_i(T_1)\ell_i(T_2)},$$

where the final step is the inequality of arithmetic and geometric means. If, in turn, we can do this for *all* the colors, we conclude that

$$\Pi(T) = \prod_{i=1}^{q} \ell_i(T) \geqslant \prod_{i=1}^{q} 2\sqrt{\ell_i(T_1)\ell_i(T_2)} \geqslant 2^q \cdot \Pi_q(N/2),$$

where $\Pi_q(N/2)$ is our inductive lower bound on $\Pi(T')$ for every q-edge-colored transitive tournament T' on $N/2$ vertices. Inductively, we know that $\Pi_q(N/2) \geqslant (N/2)^{q-o(q)}$, hence we learn that $\Pi(T) \geqslant 2^q(N/2)^{q-o(q)} \geqslant N^{q-o(q)}$. That is, in the dream scenario, we can prove the desired result.

Of course, there is no reason for the dream scenario to be remotely close to true—we shouldn't even expect to be able to glue together a longest i-avoiding path in each half even for *a single* color i, let alone for all of them. The proof now consists of showing that either we can glue together *almost* longest i-avoiding paths for *almost* all colors i, or else that we win for another reason; the losses inherent in these "almosts" are what contribute to the error term in the exponent.

We begin by explaining the first "almost". Because we are allowed to tolerate errors, it is not necessary for P_1 and P_2 above to be truly the longest i-avoiding paths in T_1, T_2, respectively. Instead, it suffices that their lengths are nearly as long as possible, so that $\ell_i(T) \geqslant (2 - o(1))\sqrt{\ell_i(T_1)\ell_i(T_2)}$. This gives us a lot more flexibility, since it now suffices to just find one such pair P_1, P_2, of nearly maximal length, such that the edge joining the end of P_1 and the start of P_2 is not colored with color i. To accomplish this, we will consider a large number of nearly maximal i-avoiding paths in each half, and attempt to glue together every pair.

Concretely, set $s = \varepsilon N/q$, where $\varepsilon > 0$ is a small parameter depending only on q (and tending to 0 as $q \to \infty$). We wish to pick s different i-avoiding paths in each half; moreover, since our goal is to try gluing every pair of paths from each half, we want these s paths to have different endpoints. To accomplish this, let us define $\ell_i({}^{\to}v)$ to be the length of the longest i-avoiding directed path ending at v, and similarly $\ell_i(w^{\to})$ to be the length of the longest i-avoiding directed path starting at w. We now define X_i to be the set of the s vertices in the first half with the highest value of $\ell_i({}^{\to}\bullet)$. Similarly, we define Y_i to be the set of the s vertices in the second half with the largest value of $\ell_i(\bullet^{\to})$. The paths P_1 that we consider will then be the longest i-avoiding paths ending at a vertex of X_i, and similarly the P_2 we consider are the longest i-avoiding paths starting at a vertex of Y_i. Heuristically, since $s \ll N$ and X_i consists of the s "best" endpoints for i-avoiding paths in T_1, any path ending at a vertex of X_i is the endpoint of such a path of length almost $\ell_i(T_1)$.

We now call a color i *compressed* if the majority of the vertices of X_i and the majority of the vertices of Y_i are close to the midpoint $N/2$, i.e. in the interval $[\frac{N}{2} - 4s, \frac{N}{2} + 4s]$. The claim now is that we can glue together i-avoiding paths in T_1 and T_2 for nearly all compressed colors. Indeed, if i is a compressed color and we cannot perform such a gluing, that means that all edges from X_i to Y_i are of color i, and in particular we have found many edges of color i within the interval $[\frac{N}{2} - 4s, \frac{N}{2} + 4s]$. As there are only $O(s^2)$ edges within this interval, we must be able to glue for the vast majority of compressed colors: each color in which we cannot contributes many edges in this interval, and there are not so many such edges.

Thus, if nearly all colors are compressed, we are "almost" in the dream scenario and we can prove the claim inductively by absorbing all the losses in the $N^{-o(q)}$ factor. It remains to understand what happens when we have a large number, say $2p$, of non-compressed colors; by symmetry, we may assume that at least p of them are *left-diffuse*, i.e. that the majority of X_i for these colors lies in the interval $[1, \frac{N}{2} - 4s]$. Let $U_i \subseteq X_i$ be the vertices far from the midpoint, for each of these p left-diffuse colors i.

The key claim now is the following. Suppose that v, w are vertices, where v precedes w in the ordering of $[N/2]$, and assume that $v \in X_i, w \notin X_i$. Then the edge vw is necessarily colored with color i. Indeed, if it were not, then we can extend any i-avoiding path ending at v to a longer one ending at w, but then the fact that $v \in X_i, w \notin X_i$ contradicts the definition of X_i. This simple observation has two immediate corollaries. First, if i, j are distinct left-diffuse colors, then U_i and U_j are disjoint. Indeed, since there are few vertices of $X_i \cup X_j$ in $[\frac{N}{2} - 4s, \frac{N}{2}]$, we can find $w \notin X_i \cup X_j$ in this interval. If $v \in U_i \cap U_j$, then the observation above implies that vw must receive both color i and color j, which is impossible, hence $U_i \cap U_j = \varnothing$. The second corollary is that the color of every edge between U_i and U_j must be left-diffuse; indeed, by the key claim, every such edge must in fact be either of color i or of color j.

This immediately has the following remarkable consequence. For every color k which is *not* one of our p left-diffuse colors, we have

$$\ell_k(T) \geqslant \sum_{i \text{ left-diffuse}} \ell_k(T[U_i]) \geqslant p \left(\prod_{i \text{ left-diffuse}} \ell_k(T[U_i]) \right)^{1/p}.$$

Indeed, we get the desired k-avoiding path by using the vertices from the longest k-avoiding path in each U_i. This path is k-avoiding as every edge of the path is either between vertices from the same U_i or must be one of the p left-diffuse colors. The last inequality is the arithmetic mean-geometric mean inequality. In other words, we find that for $q-p$ choices of color k, we essentially gain a factor of p in the length of the longest k-avoiding path. This is an enormous gain, much larger than the factor of 2 we were winning in the dream scenario; moreover, by choosing $p = o(q)$, we can obtain such a gain for nearly all colors. The loss comes from the fact that we now need to apply the inductive hypothesis to the subtournament induced on $\bigcup_{i \text{ left-diffuse}} U_i$, which is quite small; however, by picking p appropriately, we can ensure that the amount we win by multiplying the lengths by p dominates this loss.

Apart from the computations, the discussion above is an essentially complete sketch of the proof of Theorem 1.6 in the case that T is transitive. We now discuss the difficulties encountered when passing to the setting of general tournaments, and the main additional ideas needed to overcome them.

First, by applying Theorem 1.7, we may assume that T is δ-close to transitive, for some small constant $\delta = \delta(q) > 0$. Indeed, if this is not the case, then Theorem 1.7 shows that T contains a path of linear length which receives at most 3 colors, a far stronger statement than what we aim to prove. We now fix an ordering of $V(T)$ such that all but at most δN^2 edges go forward. In fact, by deleting a small number of vertices, we can even pass to a "minimum degree" version of this statement: for every vertex, at most δN of its out-neighbors precede it in the ordering, and at most δN of its in-neighbors follow it in the ordering. At this point, most of the argument presented in the transitive case goes through: while there may be a small number of edges that we cannot use, the arguments are sufficiently robust to overcome this.

However, there is a major thing that stops working, which is instrumental to using the key claim above. Namely, if v precedes w in the ordering, and even if $v \to w$ in T, we may have that vw does not receive color i, and yet not be able to conclude that $\ell_i(w) > \ell_i(v)$. Indeed, it is possible that all long i-avoiding paths ending at v have already passed through w, and hence we cannot simply extend such a path to end at w. This is a fundamental issue[2] which is not present in the fully transitive setting: there, a directed path must be monotone, and hence cannot have already used a "future" vertex.

In order to overcome this issue, we split the set of colors into *long* colors—those colors i where there is an i-avoiding path of length at least γN, for some small $\gamma > 0$—and *short* colors. For short colors, the arguments used above can now be made to work: the longest i-avoiding path ending

[2]We remark that this issue is quite similar to the main difficulty in proving the Gallai–Hasse–Roy–Vitaver theorem.

at w has length at most γN, hence there are relatively few vertices w for which we cannot argue about the color of the edge vw, as most vertices w are not on such a path. Dealing with these "exceptional" vertices now requires some care, but no genuinely new ideas. On the other hand, we know essentially nothing about the long colors, except that they are long. If we only aimed to prove that T contains a long color-avoiding directed path, then long colors would immediately help us; unfortunately, we strengthened the inductive hypothesis to argue about $\Pi(T)$, so we cannot simply say we are done if there is a long color.

In order to overcome this issue, the final idea is to *weaken* the inductive hypothesis. Rather than inductively studying the function $\Pi(T)$ defined above, we will instead argue inductively about the modified function

$$\prod_{i=1}^{q} \min\{\ell_i(T), \gamma N\}.$$

For the short colors, the term controlling the minimum is $\ell_i(T)$, hence we can use the same argument as before. For long colors, the main term is γN, and luckily this quantity is trivial to handle inductively: for example, since $\gamma N = \gamma \frac{N}{2} + \gamma \frac{N}{2}$, we can automatically obtain the doubling we seek in the dream scenario for long colors. With these modifications, the rest of the proof goes through essentially as before.

3. Simple proofs

3.1. Proof of Theorem 1.7.

Proof of Theorem 1.7. The first two authors [20, Lemma 1.3] proved the following quantitative variant of the triangle removal lemma in tournaments: There is a constant $c > 0$ such that if a tournament on N vertices is δ-far from transitive, then it contains at least $c\delta^2 N^3$ cyclic triangles. For each cyclic triangle, arbitrarily label the edges cyclically as $1, 2, 3$. Since there are q colors, there are q^3 possible colorings of the edges of a cyclic triangle with the edges labeled as $1, 2, 3$. By the pigeonhole principle, there are at least $c\delta^2 N^3/q^3$ cyclic triangles whose labeled edges have the same color pattern (call this pattern P). Greedily delete edges from the tournament that are in fewer than $2c\delta^2 N/q^3$ such cyclic triangles of color pattern P. As there are fewer than $N^2/2$ edges, we still have more than $c\delta^2 N^3/q^3 - (N^2/2)(2c\delta^2 N/q^3) > 0$ cyclic triangles of color pattern P remaining after we are done with this process. Each edge that remains (and there is at least one since there is at least one cyclic triangle remaining) extends to at least $2c\delta^2 N/q^3$ cyclic triangles of color pattern P. Starting with the vertices v_1, v_2, v_3 of one remaining cyclic triangle of color pattern P with $v_1 \to v_2$, we can greedily build the desired directed path (and furthermore the square of a path with the desired properties) by adding, after v_{i-1}, v_i, a vertex v_{i+1} not already on the path such that v_{i-1}, v_i, v_{i+1} forms a cyclic triangle of color pattern P in what remains. We can necessarily find such a vertex v_{i+1} as long as $2c\delta^2 N/q^3 > i - 1$ since each edge on the path extends to more than $2c\delta^2 N/q^3$ cyclic triangles of color pattern P in what remains. Hence, the path length we get in the end is at least $2c\delta^2 N/q^3$. □

3.2. General reductions and bounds.

If K is a q-edge-colored ordered complete graph, we denote by $f_{q,r}(K)$ the maximum length of a monotone path in K which is colored by at most r colors. We then let $f_{q,r}(N)$ be the minimum of $f_{q,r}(K)$ over all q-edge-colored ordered N-vertex complete graphs K. The following simple observation, which already appears in [22, 26] captures the fact that when $r = q - 1$, this notion is equivalent to the study of $(q-1)$-increasing sequences in $[n]^q$.

Proposition 3.1. *We have $f_{q,q-1}(N) \leqslant n$ if and only if $F_{q,q-1}(n) \geqslant N$.*

Proof. First, suppose that $f_{q,q-1}(N) \leqslant n$. Thus, there exists some q-edge-colored ordered clique K on vertex set $[N]$ in which every monotone path with at most $q-1$ colors contains at most n

vertices. We define vectors $x_1, \dots, x_N$ by letting x_a record, for each $i \in [q]$, the length of the longest monotone path ending at a which does not use color i. Note that we are measuring path length by the number of vertices, hence each entry of each vector x_a is at least 1. Moreover, by assumption, every color-avoiding monotone path in K has length at most n, so we conclude that $x_a \in [n]^q$ for all $a \in [N]$. We now claim that for all $a < b$, we have that $x_a <_{q-1} x_b$. Indeed, if the color of ab is i, then for every $j \neq i$, we can extend every j-avoiding monotone path ending at x_a by one to be a j-avoiding monotone path ending at x_b. Hence, every coordinate of x_b, except potentially the ith coordinate, is strictly larger than the corresponding coordinate of x_a. We conclude that $F_{q,q-1}(n) \geqslant N$, as desired.

For the converse, let's suppose that $F_{q,q-1}(n) \geqslant N$, and fix vectors $x_1, \dots, x_N \in [n]^q$ witnessing this. We construct a q-edge-coloring of the complete graph on vertex set $[N]$ as follows: for every $a < b$, we know that $x_a <_{q-1} x_b$, hence there is at most one coordinate i for which $(x_a)_i \geqslant (x_b)_i$. If there is such an i, we color the edge ab with color i; if not, we color it arbitrarily. The key property of this coloring is that if ab is *not* colored i, then $(x_a)_i < (x_b)_i$ (for if this were not the case, we would be forced to color the edge i by the rule above). We claim that this coloring has no monotone color-avoiding path of length $n+1$, proving that $f_{q,q-1}(N) \leqslant n$. Indeed, suppose for contradiction that $v_0, v_1, \dots, v_n$ form a monotone color-avoiding path in this coloring, say avoiding the color i. Since each consecutive pair is not colored i, we must have that $(x_{v_a})_i < (x_{v_{a+1}})_i$ for all $0 \leqslant a < n$. But the numbers $(x_{v_0})_i, \dots, (x_{v_n})_i$ all lie in $[n]$, so it is not possible for these $n+1$ numbers to be strictly increasing. This contradiction completes the proof. $\square$

It is worth remarking that the same argument above shows that $F_{q,r}(n)$ is equal to the largest N for which the following holds. There is a complete ordered graph on N vertices in which each edge is assigned a subset of r colors from a set of q colors, and no color appears in the color set of every edge of a monotone path P of length $n+1$. This is one less than the *set-coloring ordered Ramsey number* of a monotone path with $n+1$ vertices. Set-coloring Ramsey numbers have been extensively studied recently for cliques (see [2, 10, 12]) and have found interesting connections to coding theory, and have also been studied for certain other classes of graphs (see e.g. [7] and the references therein). Ordered Ramsey numbers of graphs have also been extensively studied recently (see [5, 11] and the recent survey [4]).

Similarly, $G_{q,r}(n)$ is at most the largest N for which the following holds. There is a tournament on N vertices in which each edge is assigned a subset of r colors from a set of q colors, and no color appears in the color set of every edge of a directed path of length $n+1$. This is one less than the set-coloring oriented Ramsey number of a path with $n+1$ vertices with parameters (q,r). Directed Ramsey numbers have similarly received much recent attention (see, for instance, [8, 18]). It would be interesting to study set-coloring Ramsey numbers for other graphs, ordered graphs, and acyclic digraphs.

Another useful observation is that monotone paths behave very cleanly under lexicographic product. Recall that an *ordered graph* is a graph equipped with a linear order on its vertex set. Given two ordered graphs G_1, G_2, we define their *lexicographic product* $G_1 \otimes G_2$ by first duplicating each vertex of G_2 to $|G_1|$ vertices, and replacing each edge of G_2 by a complete bipartite graph between these blowup sets. We then insert a copy of G_1 into each of these blowup sets, while otherwise maintaining the order. That is, each original vertex of G_2 now corresponds to an interval of length $|G_1|$ in the new ordering.

Proposition 3.2. *Let G_1, G_2 be ordered graphs, and let L_1, L_2 be the length of the longest monotone path in G_1, G_2, respectively. Then the length of the longest monotone path in $G_1 \otimes G_2$ is exactly $L_1 L_2$.*

Proof. The lower bound is immediate: we can replace each vertex of the longest monotone path in G_2 by the longest monotone path in G_1, thus obtaining a monotone path in $G_1 \otimes G_2$ of length $L_1 L_2$. For the upper bound, fix any monotone path in $G_1 \otimes G_2$. Note that this path can visit at most L_2 of the blowup parts, since every time it moves between blowup parts, it needs to use an edge from the blowup of G_2. Thus, under the natural projection to G_2, it traverses a monotone path in G_2, hence it visits at most L_2 parts. But within each part, it simply traverses a monotone path in G_1, hence it uses at most L_1 vertices in each part. In total, the number of vertices visited is at most $L_1 L_2$. $\square$

As an immediate corollary, we obtain the same result for paths with restricted colors. Let K_1 and K_2 be q-edge-colored ordered complete graphs (possibly on different numbers of vertices). We define their *lexicographic product* $K_1 \otimes K_2$ in the same way as above: we duplicate each vertex of K_2 to $|K_1|$ vertices, replace each edge of K_2 by a complete bipartite graph of the same color, and insert a copy of K_1 into each of these blowup sets, while otherwise maintaining the order. If $S \subseteq [q]$ is a set of colors, we say that a path is *S-colored* if it is colored only by colors in S.

Proposition 3.3. *Let K_1, K_2 be two q-edge-colored ordered complete graphs, and let $S \subseteq [q]$ be an arbitrary set of colors. Let L_1, L_2 be the length of the longest monotone S-colored path in K_1, K_2, respectively. Then the length of the longest S-colored monotone path in $K_1 \otimes K_2$ is exactly $L_1 L_2$.*

Proof. Let G_1, G_2 be the ordered graphs comprising all edges in K_1, K_2, respectively, colored by a color in S. Then $G_1 \otimes G_2$ is precisely the set of edges in $K_1 \otimes K_2$ receiving a color in S. Thus, the claimed result follows immediately from Proposition 3.2. $\square$

One consequence of Proposition 3.3 is the upper bound $f_{q,r}(N) \leqslant N^{r/q}$ for all N which are a power of q. Indeed, let $N = m^q$, and let $K_1, \ldots, K_q$ be ordered cliques on m vertices, where all edges in K_i are given color i. By Proposition 3.3, for every $S \subseteq [q]$, the length of the longest S-colored monotone path in $K = K_1 \otimes \cdots \otimes K_q$ is precisely $m^{|S|}$; in particular, the longest monotone path in K receiving at most r colors has length exactly $m^r = (m^q)^{r/q}$, proving the bound $f_{q,r}(N) \leqslant N^{r/q}$ for $N = m^q$.

Another immediate corollary of Proposition 3.3 is that the extremal function $f_{q,r}(N)$ that we are studying is sub-multiplicative.

Corollary 3.4. *For all integers N_1, N_2, q, r, we have $f_{q,r}(N_1) f_{q,r}(N_2) \geqslant f_{q,r}(N_1 N_2)$.*

Proof. Let K_1, K_2 be q-edge-colored ordered complete graphs on N_1, N_2 vertices, respectively, with the property that every monotone path in K_i which uses at most r colors has length at most $f_{q,r}(N_i)$. Let $K = K_1 \otimes K_2$, so that $f_{q,r}(N_1 N_2) \leqslant f_{q,r}(K)$. On the other hand, for every set $S \subseteq [q]$ of exactly r colors, we know from Proposition 3.3 that the maximum length of an S-colored monotone path in K is at most $f_{q,r}(K_1) f_{q,r}(K_2)$, implying that $f_{q,r}(N_1 N_2) \leqslant f_{q,r}(K) \leqslant f_{q,r}(K_1) f_{q,r}(K_2) = f_{q,r}(N_1) f_{q,r}(N_2)$, as claimed. $\square$

One simple consequence of Corollary 3.4 is that the limiting behavior of $f_{q,r}(N)$ is as a power of N, as stated in the following proposition.

Proposition 3.5. *For every $q > r \geqslant 1$, there exists some $\alpha \in [0, 1]$ such that $f_{q,r}(N) = N^{\alpha + o(1)}$ as $N \to \infty$.*

Proof. Fix $q > r \geqslant 1$, and define $s_k = \log(f_{q,r}(2^k))$ for every integer $k \geqslant 0$. Then the sequence $(s_k)_{k \geqslant 1}$ is sub-additive, since Corollary 3.4 shows that

$$s_{k_1+k_2} = \log(f_{q,r}(2^{k_1+k_2})) \leqslant \log(f_{q,r}(2^{k_1}) f_{q,r}(2^{k_2})) = \log(f_{q,r}(2^{k_1})) + \log(f_{q,r}(2^{k_2})) = s_{k_1} + s_{k_2}$$

for all integers $k_1, k_2 \geqslant 0$. Recall that Fekete's lemma states that for every sub-additive sequence (s_k), the limit $\lim_{k \to \infty} s_k / k$ exists (but may be $-\infty$). Let us denote this limit by α. Note that since $f_{q,r}(N) \geqslant 1$ for all N, we have that $s_k \geqslant 0$ for all k, and therefore $\alpha \geqslant 0$. Similarly, since $f_{q,r}(N) \leqslant N$ for all N, we have $s_k \leqslant k$ for all k, and thus $\alpha \leqslant 1$.

Finally, we note that $s_k/k = \alpha + o(1)$, or equivalently $s_k = \alpha k + o(k)$, or equivalently $f_{q,r}(2^k) = 2^{\alpha k + o(k)} = (2^k)^{\alpha + o(1)}$. This proves the claim in case N is of the form 2^k for some integer k. For all other N, we argue as follows: if $k = \lfloor \log N \rfloor$, so that $2^k \leqslant N < 2^{k+1}$, then we have $2^k = N^{1+o(1)}$ and $2^{k+1} = N^{1+o(1)}$. Since the function $f_{q,r}$ is monotone, we have

$$N^{\alpha+o(1)} = (2^k)^{\alpha+o(1)} = f_{q,r}(2^k) \leqslant f_{q,r}(N) \leqslant f_{q,r}(2^{k+1}) = (2^{k+1})^{\alpha+o(1)} = N^{\alpha+o(1)}. \qquad \square$$

As another consequence of Corollary 3.4, we can prove a "finitary" version of this limiting statement, similar to the use of the tensor product trick in the study of Sidorenko's conjecture. It shows that whenever we prove a lower bound of the form $f_{q,r}(N) \geqslant N^{\alpha-o(1)}$ for some α and some $o(1)$ term tending to zero as $N \to \infty$, we can automatically deduce the stronger bound $f_{q,r}(N) \geqslant N^{\alpha}$ for all N.

Proposition 3.6. *Suppose that for all N, we have the estimate $f_{q,r}(N) \geqslant N^{\alpha-\varepsilon(N)}$, for some function $\varepsilon(N)$ with $\lim_{N\to\infty} \varepsilon(N) = 0$. Then, in fact, we have $f_{q,r}(N) \geqslant N^{\alpha}$ for all N.*

Proof. Suppose for contradiction that $f_{q,r}(N) < N^{\alpha}$ for some N. In particular, we may write $f_{q,r}(N) = N^{\alpha-\delta}$ for some constant $\delta > 0$. By Corollary 3.4, for every $t \geqslant 1$, we have

$$f_{q,r}(N^t) \leqslant f_{q,r}(N)^t = (N^{\alpha-\delta})^t = (N^t)^{\alpha-\delta}.$$

Now, we pick t sufficiently large so that $\varepsilon(N^t) < \delta$, which contradicts our assumption that $f_{q,r}(N^t) \geqslant (N^t)^{\alpha-\varepsilon(N^t)}$. $\square$

Recall from Section 2 that in our proof of Theorem 1.6, we work inductively with the quantity

$$\Pi(K) := \prod_{i=1}^{q} \ell_i(K),$$

where $\ell_i(K)$ denotes the length of the longest i-avoiding monotone path in K. A priori, working with this product may appear to be wasteful, since perhaps the longest color-avoiding path is much longer than the geometric mean of all of them. As an immediate consequence of Proposition 3.3, we can show that this is not the case, and that the two notions are equivalent for large N.

Proposition 3.7. *Let K be an N-vertex q-edge-colored ordered complete graph. There is a q-edge-colored ordered clique $\widetilde{K}$ on N^q vertices for which $\ell_i(\widetilde{K}) = \Pi(K)$ for all $i \in [q]$.*

Proof. Let $K_1, \dots, K_q$ be obtained from K by cyclically permuting the colors q times. That is, $K_1 = K$, and K_t is obtained from K_{t-1} by recoloring each edge of color i to color $i + 1 \pmod q$. Let $\widetilde{K} = K_1 \otimes \cdots \otimes K_q$, so that $|\widetilde{K}| = N^q$. By Proposition 3.3, we know that

$$\ell_i(\widetilde{K}) = \prod_{t=1}^{q} \ell_i(K_t).$$

But K_t is obtained from K by cyclically permuting the colors, hence $\prod_{t=1}^{q} \ell_i(K_t) = \prod_{i=1}^{q} \ell_i(K) = \Pi(K)$, as claimed. $\square$

For a q-edge-colored tournament T, we denote by $g_{q,r}(T)$ the maximum length of a directed path in T that receives at most r colors. We then let $g_{q,r}(N)$ denote the minimum of $g_{q,r}(T)$ over all N-vertex tournaments T. Note that we have the trivial bound $g_{q,r}(N) \leqslant f_{q,r}(N)$, since $f_{q,r}(N)$ precisely corresponds to restricting this minimization to *transitive* tournaments. The following lemma records some simple relations between these functions for different choices of the parameters r and q. These bounds can be seen as analogues of the simple relations (2) for the vector functions $F_{q,r}$ and $G_{q,r}$.

Lemma 3.8. *Let $q > r \geqslant 1$ and $N \geqslant 1$ be integers.*

(1) *We have that $f_{q,r}(N) \geqslant f_{q-t,r-t}(N)$ and $g_{q,r}(N) \geqslant g_{q-t,r-t}(N)$ for all $0 \leqslant t < r$.*
(2) *We have $f_{q,r}(N) \geqslant f_{q/d,r/d}(N)$ and $g_{q,r}(N) \geqslant g_{q/d,r/d}(N)$ for all d dividing* $\gcd(q, r)$.
(3) *If $p = \lfloor \frac{q}{q-r} \rfloor \geqslant 2$, we have $f_{q,r}(N) \geqslant f_{p,p-1}(N)$ and $g_{q,r}(N) \geqslant g_{p,p-1}(N)$.*

Proof. Let T be a q-edge-colored N-vertex tournament. For (1), arbitrarily select $t+1$ colors and merge them into a single color, thus obtaining a $(q-t)$-edge-colored tournament. Any directed path in this auxiliary coloring which receives at most $r-t$ colors corresponds to a directed path in T receiving at most r colors, proving that $g_{q,r}(N) \geqslant g_{q-t,r-t}(N)$. As this argument does not change the tournament, we can restrict it to transitive tournaments to obtain the claim for $f_{q,r}$.

For (2), we arbitrarily group the q colors into q/d sets of size d, and merge the colors in each set. We obtain a (q/d)-edge-colored tournament, in which every directed path receiving at most r/d colors corresponds to a path in T receiving at most r colors. This shows $g_{q,r}(N) \geqslant g_{q/d,r/d}(N)$, and applying the same argument to transitive tournaments yields the claim for $f_{q,r}$.

Finally, for (3), write $q = p(q-r)+t$ for some $0 \leqslant t < q-r$. Note that $p \geqslant 2$ implies $q \geqslant 2q-2r$, or equivalently $q-r \leqslant r$, so $0 \leqslant t < r$. We also have $q-t = p(q-r)$ and $(p-1)(q-r) = p(q-r)-(q-r) = (q-t)-(q-r) = r-t$ by definition of t. Therefore,

$$f_{q,r}(N) \geqslant f_{q-t,r-t}(N) = f_{p(q-r),(p-1)(q-r)}(N) \geqslant f_{p,p-1}(N),$$

using (1) in the first inequality and (2) in the second. The exact same argument holds for $g_{q,r}$. $\square$

Although these bounds are very simple, they are sufficient to determine $f_{q,r}$ and $g_{q,r}$ in certain special cases. For example, recall that the Erdős–Szekeres lemma implies that $f_{q,1}(N) \geqslant N^{1/q}$, and the Gallai–Hasse–Roy–Vitaver theorem implies that $g_{q,1}(N) \geqslant N^{1/q}$; both bounds are tight if N is a power of q. Using Lemma 3.8(2), we conclude that for all $d \geqslant 1$,

$$f_{qd,d}(N) \geqslant g_{qd,d}(N) \geqslant g_{q,1}(N) \geqslant N^{1/q},$$

which is again tight for N a power of q by Proposition 3.3.

3.3. **Reductions: proof of Theorems 1.2 and 1.5.** We now show how the various reductions proved in the previous subsection immediately imply Theorems 1.2 and 1.5, assuming Theorem 1.6. We begin with Theorem 1.5.

Proof of Theorem 1.5, assuming Theorem 1.6. Theorem 1.6 states that every q-edge-colored N-vertex tournament contains a color-avoiding directed path of length at least $c_q N^\alpha$, where $\alpha = 1 - O(1/\sqrt{\log q})$. In particular, applying this to transitive tournaments, we find that $f_{q,q-1}(N) \geqslant c_q N^\alpha = N^{\alpha-\varepsilon(N)}$, where $\varepsilon(N) = \log \frac{1}{c_q} / \log N$. Since $\lim_{N\to\infty} \varepsilon(N) = 0$, we may apply Proposition 3.6 to conclude that, in fact, $f_{q,q-1}(N) \geqslant N^\alpha = N^{1-O(1/\sqrt{\log q})}$ for all N. This is precisely the statement of Theorem 1.5. $\square$

Similarly, Theorem 1.2 follows immediately, as it is equivalent to Theorem 1.5.

Proof of Theorem 1.2, assuming Theorem 1.6. We have already seen how Theorem 1.6 implies Theorem 1.5, hence we may assume that $f_{q,q-1}(N) \geqslant N^\alpha$ for all N, where $\alpha = 1 - O(1/\sqrt{\log q})$. By Proposition 3.1, this is the same as saying that $F_{q,q-1}(n) \leqslant n^{1/\alpha}$. But $1/\alpha = 1 + O(1/\sqrt{\log q})$, which proves Theorem 1.2. $\square$

4. Proof of Theorem 1.6

Henceforth, whenever working with a q-coloring, we will always assume that the palette of colors is $[q]$. Given a q-colored tournament T and a color $i \in [q]$, we let $\ell_i(T)$ denote the length of the longest directed path in T which does not use the color i. Recall that we define the length of a path to be the number of vertices it comprises, so that $\ell_i(T) \geqslant 1$ for every T and every $i \in [q]$.

We define a parameter γ by

$$\gamma = \frac{1}{2q \cdot 2^{\sqrt{\log q}}}. \tag{6}$$

For an N-vertex q-edge-colored tournament T, we then define

$$m_i(T) \coloneqq \min\{\ell_i(T), \gamma N\}$$

and

$$\Pi(T) \coloneqq \prod_{i=1}^{q} m_i(T) = \prod_{i=1}^{q} \min\{\ell_i(T), \gamma N\}$$

and finally let $\Pi_q(N)$ be the minimum value of $\Pi(T)$ over all N-vertex q-colored tournaments T.

Our main technical result, which immediately implies Theorem 1.6, is the following lower bound for $\Pi_q(N)$.

Proposition 4.1. *We have that $\Pi_q(N) \geqslant c_q N^{q-Cq/\sqrt{\log q}}$, where $C > 0$ is an absolute constant and $c_q > 0$ is a constant depending only on q.*

Note that Proposition 4.1 immediately implies Theorem 1.6. Indeed, fix some q-colored tournament T on N vertices. If $\ell_i(T) \geqslant \gamma N$ for some $i \in [q]$, then we are done since $\gamma N \geqslant c_q N^{1-C/\sqrt{\log q}}$, by picking c_q to be smaller than γ. Hence we may assume that $\ell_i(T) < \gamma N$ for all i, meaning that $\Pi(T) = \prod_{i=1}^{q} \ell_i(T)$. Therefore, there exists some $i \in [q]$ for which $\ell_i(T) \geqslant \Pi(T)^{1/q} \geqslant \Pi_q(N)^{1/q}$. Hence T contains a path of length $\Pi_q(N)^{1/q} \geqslant c_q N^{1-C/\sqrt{\log q}}$ which avoids color i, as claimed in Theorem 1.6.

Before presenting the proof of Proposition 4.1, we record one more tool that we will use, namely a simple degree cleaning lemma which allows us to pass from a nearly transitive tournament to an ordered graph with high minimum degree. We recall that an *ordered graph* is a graph equipped with a linear ordering $\prec$ of its vertex set. We say that an ordered graph G is *consistent* with a tournament on the same vertex set if, for all $v \prec w$, we have that $vw \in E(G)$ if and only if $v \to w$ in T. Note that a tournament and an ordering of its vertices uniquely defines an ordered graph (with the same ordering) that is consistent with it; conversely, every ordered graph uniquely determines a tournament consistent with it.

Lemma 4.2. *Let $0 < \delta < \frac{1}{2}$, and let T_0 be an N_0-vertex tournament which is δ^2-close to transitive. Then there exists an integer $N \geqslant (1-\delta)N_0$, an N-vertex subtournament T of T_0, and an ordered N-vertex graph G with the following properties. G is consistent with T, and every vertex of G has at most $4\delta N$ non-neighbors in G.*

Proof. By assumption, we can reverse the orientation of at most $\delta^2 N_0^2$ edges in T_0 in order to obtain a transitive tournament. Let us label $V(T)$ as $w_1, \dots, w_{N_0}$ according to this transitive ordering. Thus, in T, at most $\delta^2 N_0^2$ edges are backwards with respect to this ordering, i.e. there are at most $\delta^2 N_0^2$ pairs (w_a, w_b) with $a < b$ and $w_a \leftarrow w_b$. Let H_0 be the graph of these backwards edges, i.e. the graph with vertex set $\{w_1, \dots, w_{N_0}\}$, where we set (w_a, w_b) an edge for $a < b$ if and only if $w_a \leftarrow w_b$ in T. By assumption, $e(H_0) \leqslant (\delta N_0)^2$. Therefore, H_0 has at most δN_0 vertices of degree greater than $2\delta N_0$. By deleting these vertices, we obtain a graph H on $N \geqslant N_0 - \delta N_0 = (1-\delta)N_0$ vertices with maximum degree at most $2\delta N_0 \leqslant 4\delta N$. To conclude, we set $G = \overline{H}$, and set T to be the subtournament induced on the vertices of G. $\square$

We are now ready to prove Proposition 4.1, and hence also Theorem 1.6.

Proof of Proposition 4.1. Recall that $\Pi_q(N) \geqslant 1$ for all N and q, hence the result is vacuously true if q is fixed and C is large. Thus, we can and will assume henceforth that q is at least some large absolute constant. The proof now proceeds by induction on N for each fixed q; again, by picking c_q sufficiently small, we can make the result vacuously true if N is small, hence we have the base case of our induction.

We begin by defining several parameters that we will need during the proof. We let

$$p = \frac{24q}{2^{\sqrt{\log q}}}, \qquad s = 2\gamma N = \frac{N}{q \cdot 2^{\sqrt{\log q}}}, \qquad \delta = \frac{\gamma}{8} = \frac{1}{16q \cdot 2^{\sqrt{\log q}}}. \tag{7}$$

To help the reader keep track of these, we note that in the proof, we will pick out q disjoint sets of exactly s vertices, hence we want s to be smaller, but not much smaller, than N/q. Additionally, we will eventually encounter a special set of p colors, hence we want p to be smaller, but not much smaller, than q. The precise quantities defined above are obtained by optimizing the final contributions of the various steps of the proof. It will also be useful to notice for the future that $4\delta N = s/4$.

We now fix some N_0, and assume that we have proved the result for all smaller values of N. We also fix a q-edge-colored tournament T_0 on N_0 vertices, and we aim to show that $\Pi(T_0) \geqslant c_q N_0^{q-Cq/\sqrt{\log q}}$. We first split into cases depending on whether or not T_0 is δ^2-close to transitive, where we recall that δ is defined in (7). If it is not, then we apply Theorem 1.7. We find that T_0 contains a directed path of length at least $c\delta^4 N_0/q^3$, where $c > 0$ is an absolute constant, which is colored by at most 3 colors. Without loss of generality, let us assume that these three colors are $1, 2, 3$. This means that for every $i \in [q] \setminus \{1, 2, 3\}$, we have $\ell_i(T_0) \geqslant c\delta^4 N_0/q^3$. As a consequence,

$$\Pi(T_0) = \prod_{i=1}^{q} m_i(T) \geqslant \prod_{i=4}^{q} \min\{\ell_i(T_0), \gamma N_0\} \geqslant \prod_{i=4}^{q} \min\left\{\frac{c\delta^4}{q^3}, \gamma\right\} \cdot N_0 = \left(\min\left\{\frac{c\delta^4}{q^3}, \gamma\right\}\right)^{q-3} N_0^{q-3}.$$

Since δ and γ both depend only on q, the first term is a positive constant depending only on q, and in particular can be made larger than c_q by choosing c_q appropriately. Moreover, by picking C appropriately, we can ensure that $Cq/\sqrt{\log q} \geqslant 3$ for all q, hence $N_0^{q-3} \geqslant N_0^{q-Cq/\sqrt{\log q}}$. Therefore, we conclude that $\Pi(T_0) \geqslant c_q N_0^{q-Cq/\sqrt{\log q}}$, as desired.

We may thus assume that T_0 is δ^2-close to transitive. We now apply Lemma 4.2 to pass to a subtournament T on $N \geqslant (1-\delta)N_0$ vertices, as well as an ordered graph G consistent with T in which every vertex has at most $4\delta N$ non-neighbors. As T is a subtournament of T_0, we have $\Pi(T_0) \geqslant \Pi(T)$, and therefore it suffices to prove that $\Pi(T) \geqslant c_q N_0^{q-Cq/\sqrt{\log q}}$. Moreover, since $N \geqslant (1-\delta)N_0 \geqslant 2^{-2\delta} N_0$, it in turn suffices to prove that

$$\Pi(T) \geqslant 2^{\delta q} \cdot c_q \cdot N^{q-Cq/\sqrt{\log q}}, \tag{8}$$

which will be our goal for the remainder of the proof. We recall that our inductive hypothesis is that for all $N' < N$, we have $\Pi_q(N') \geqslant c_q \cdot (N')^{q-Cq/\sqrt{\log q}}$.

We say that a color $i \in [q]$ is *long* if $\ell_i(T) \geqslant \gamma N$, and we say that it is *short* otherwise. Note that for any color $i \in [q]$, we have

$$m_i(T) = \min\{\ell_i(T), \gamma N\} = \begin{cases} \ell_i(T) & \text{if } i \text{ is short,} \\ \gamma N & \text{if } i \text{ is long.} \end{cases}$$

The long colors play a special role in our argument, and must be dealt with separately. In most of what follows, we argue almost exclusively about the short colors.

As G is an N-vertex ordered graph, we henceforth identify the vertices of G with the interval $[N]$, so that the ordering $\prec$ of G agrees with the standard ordering of N. We alternate between these two notations as convenient. We begin by partitioning $[N] = V(G) = V(T)$ into four intervals A, B, C, D, of lengths $\frac{N}{2} - 4s, 4s, 4s, \frac{N}{2} - 4s$, respectively, where we recall that s was defined in (7). That is, we have

$$A = [1, \tfrac{N}{2} - 4s], \qquad B = (4s, \tfrac{N}{2}], \qquad C = (\tfrac{N}{2}, \tfrac{N}{2} + 4s], \qquad D = (\tfrac{N}{2} + 4s, N].$$

For a vertex $v \in A \cup B$ and a color $i \in [q]$, we denote by $\ell_i(^{\rightarrow}v)$ the length of the longest directed path in $T[A \cup B]$ which ends at the vertex v and whose edges avoid the color i. Note that we are restricting only to those directed paths which are entirely contained in $A \cup B$. Similarly, for $v \in C \cup D$, we denote by $\ell_i(v^{\rightarrow})$ the length of the longest i-avoiding directed path in $T[C \cup D]$ that starts at the vertex v.

For every color $i \in [q]$, we let X_i consist of the s vertices in $A \cup B$ which have the largest values of $\ell_i(^{\to}v)$. That is, for every vertex $v \in A \cup B$, we compute the integer $\ell_i(^{\to}v)$, we rank the vertices of $A \cup B$ according to these integers, and then let X_i comprise the s top-most vertices in this ranking. If there are ties in the ranking, we break them arbitrarily. Similarly, we define Y_i to comprise the s vertices in $C \cup D$ with the largest values of $\ell_i(v^{\to})$. By definition, $|X_i| = |Y_i| = s$ for all i.

Let $0 \leqslant r \leqslant q$ be the number of long colors in T. We split the $q - r$ short colors into several further types, as follows. Let i be a short color. We call the color i *left-condensed* if $|X_i \cap B| \geqslant s/2$, and *left-diffuse* if $|X_i \cap B| < s/2$. Similarly, i is *right-condensed* if $|Y_i \cap C| \geqslant s/2$, and *right-diffuse* otherwise. Finally, we call color i *condensed* if it is both left-condensed and right-condensed. We stress that we only apply these terms to short colors; in particular, whenever we speak of a condensed or left-diffuse color in what follows, that color will always also be short. We now split into cases depending on the number of condensed colors, recalling that there are exactly r long colors and $q - r$ short colors.

Case 1: There are at least $q - r - 2p$ condensed colors. Let $\Lambda \subseteq [q]$ denote the set of long colors, and let $\Xi \subseteq [q] \setminus \Lambda$ denote the set of condensed colors. Recall that every vertex of G has at most $4\delta N = s/4$ non-neighbors. As a consequence, for every $i \in \Xi$, the number of edges of G between $X_i \cap B$ and $Y_i \cap C$ is at least

$$\sum_{v \in X_i \cap B} \left(|Y_i \cap C| - \frac{s}{4}\right) \geqslant \sum_{v \in X_i \cap B} \left(\frac{s}{2} - \frac{s}{4}\right) = \frac{s}{4}|X_i \cap B| \geqslant \frac{s^2}{8},$$

where we use that color i is right-condensed in the first inequality, and that it is left-condensed in the final inequality.

Let us call a color $i \in \Xi$ *useless* if all edges of G between $X_i \cap B$ and $Y_i \cap C$ are of color i, and *useful* otherwise. By the previous computation, if i is useless, then there are at least $s^2/8$ color-i edges between B and C. Since there are at most $|B||C| = 16s^2$ edges of G between B and C, we conclude that the number of useless colors is at most $(16s^2)/(s^2/8) = 128$.

By definition, for every useful color i, there is at least one edge $v_i w_i \in E(G)$, with $v_i \in X_i \cap B$ and $w_i \in Y_i \cap C$, which does not receive color i (for if there were no such edge, then i would be useless). As G is consistent with T and B precedes C in the ordering, we know that the edge $v_i w_i$ is directed as $v_i \to w_i$. These facts imply that T contains an i-avoiding directed path of length $\ell_i(^{\to}v_i) + \ell_i(w_i^{\to})$, obtained by taking the longest i-avoiding directed path in $T[A \cup B]$ ending at v_i, concatenating it with the edge $v_i \to w_i$, and then concatenating that with the longest i-avoiding directed path in $T[C \cup D]$ starting at w_i. This is indeed a directed path since $v_i \to w_i$ and since the two paths we are concatenating have disjoint vertex sets; additionally, since the edge $v_i w_i$ does not receive color i, this is indeed an i-avoiding directed path.

Let T' be the subtournament of T induced on the vertex set $(A \cup B) \setminus \bigcup_{i=1}^q X_i$. Note that T' has at least $N' = \frac{N}{2} - sq$ vertices. Moreover, for the special vertex v_i found above, we claim that $\ell_i(^{\to}v_i) \geqslant \ell_i(T')$. Indeed, the longest i-avoiding directed path in T' is entirely contained in $V(T') \subseteq A \cup B$, and ends at some vertex $u \in V(T')$, which in particular does not belong to X_i, as X_i is disjoint from $V(T')$. As X_i comprises the s vertices with the highest values of $\ell_i(^{\to}\bullet)$, we see that $\ell_i(^{\to}v_i) \geqslant \ell_i(^{\to}u) = \ell_i(T')$. Similarly, if we let T'' be the subtournament induced on $(C \cup D) \setminus \bigcup_{i=1}^q Y_i$, then $\ell_i(w_i^{\to}) \geqslant \ell_i(T'')$ for each $i \in [q]$.

Now, for each useful color $i \in \Xi$, we have that

$$\ell_i(T) \geqslant \ell_i(^{\to}v_i) + \ell_i(w_i^{\to}) \geqslant \ell_i(T') + \ell_i(T'') \geqslant 2\sqrt{\ell_i(T')\ell_i(T'')},$$

where the final bound is the inequality of arithmetic and geometric means. Since every such color is short, we also know that $m_i(T) = \min\{\ell_i(T), \gamma N\} = \ell_i(T)$, hence

$$m_i(T) = \ell_i(T) \geqslant 2\sqrt{\ell_i(T')\ell_i(T'')} \geqslant 2\sqrt{m_i(T')m_i(T'')}$$

for every useful $i \in \Xi$. Similarly, for every long color $i \in \Lambda$, we have that $m_i(T) = \min\{\ell_i(T), \gamma N\} = \gamma N$, hence

$$m_i(T) = \gamma N \geqslant \gamma|T'| + \gamma|T''| \geqslant m_i(T') + m_i(T'') \geqslant 2\sqrt{m_i(T')m_i(T'')}.$$

In other words, we have found that for at least $q-2p-128$ of the colors i (namely the useful or long colors), the inequality $m_i(T) \geqslant 2\sqrt{m_i(T')m_i(T'')}$ holds. The remaining $2p+128$ colors are short, hence satisfy $m_i(T) = \ell_i(T)$. Moreover, for these remaining colors, we trivially have the inequality

$$m_i(T) = \ell_i(T) \geqslant \max\{\ell_i(T'), \ell_i(T'')\} \geqslant \sqrt{\ell_i(T')\ell_i(T'')} \geqslant \sqrt{m_i(T')m_i(T'')}.$$

Putting this all together, we find that

$$\Pi(T) = \prod_{i=1}^{q} m_i(T) \geqslant 2^{q-2p-128} \prod_{i=1}^{q} \sqrt{m_i(T')m_i(T'')} = 2^{q-2p-128}\sqrt{\Pi(T')\Pi(T'')}.$$

Finally, recall that T' and T'' are both q-edge-colored tournaments on at least $N' = \frac{N}{2} - sq$ vertices. By the definition of $\Pi_q(N')$, this shows that $\Pi(T'), \Pi(T'') \geqslant \Pi_q(N')$. In conclusion, we have that

$$\Pi(T) \geqslant 2^{q-2p-128}\Pi_q(N').$$

By the inductive hypothesis, we know that $\Pi_q(N') \geqslant c_q(N')^{q-Cq/\sqrt{\log q}}$. Additionally, by the definition of s in (7), we know that $N' = (\frac{1}{2} - \frac{1}{2^{\sqrt{\log q}}})N$. Since we may assume that q is sufficiently large, we have that $2^{\sqrt{\log q}} \geqslant \sqrt{\log q}$, and thus $N' \geqslant (\frac{1}{2} - \frac{1}{\sqrt{\log q}})N$. Therefore,

$$\frac{(N')^{q-Cq/\sqrt{\log q}}}{N^{q-Cq/\sqrt{\log q}}} \geqslant \left(\frac{1}{2} - \frac{1}{\sqrt{\log q}}\right)^{q-Cq/\sqrt{\log q}} = 2^{-q+Cq/\sqrt{\log q}}\left(1 - \frac{2}{\sqrt{\log q}}\right)^{q-Cq/\sqrt{\log q}}.$$

Using the inequality $1-x \geqslant 2^{-2x}$, valid for all $x \in [0, \frac{1}{2}]$ (and recalling that q is at least a sufficiently large constant, so that we may apply this inequality), we have that

$$\left(1 - \frac{2}{\sqrt{\log q}}\right)^{q-Cq/\sqrt{\log q}} \geqslant \left(1 - \frac{2}{\sqrt{\log q}}\right)^{q} \geqslant 2^{-4q/\sqrt{\log q}}.$$

Combining these bounds, we conclude that

$$\begin{aligned}
\frac{\Pi(T)}{c_q N^{q-Cq/\sqrt{\log q}}} &\geqslant \frac{2^{q-2p-128}\Pi_q(N')}{c_q N^{q-Cq/\sqrt{\log q}}} \\
&\geqslant 2^{q-2p-128} \cdot \frac{(N')^{q-Cq/\sqrt{\log q}}}{N^{q-Cq/\sqrt{\log q}}} \\
&\geqslant 2^{q-2p-128} \cdot 2^{-q+Cq/\sqrt{\log q}} \cdot 2^{-4q/\sqrt{\log q}} \\
&= 2^{(C-4)q/\sqrt{\log q}-2p-128}.
\end{aligned}$$

Finally, we recall from (7) that $p = 24q/2^{\sqrt{\log q}} \leqslant q/\sqrt{\log q}$, where the final inequality holds since q is sufficiently large. Therefore, the exponent above is at least $(C-6)q/\sqrt{\log q} - 128$. In particular, as C is sufficiently large, we have that this exponent is at least $2q/\sqrt{\log q} \geqslant 2\delta q$ for all q. We conclude that $\Pi(T) \geqslant 2^{2\delta q} \cdot c_q \cdot N^{q-Cq/\sqrt{\log q}}$. This is precisely the inequality (8) that we set out to prove, and thus this completes the proof in Case 1.

Case 2: There are fewer than $q - r - 2p$ condensed colors. In this case, there are at least $2p$ non-condensed colors, implying that there are at least p left-diffuse colors or at least p right-diffuse colors. The two cases are handled identically (in fact they are equivalent upon reversing the orientation of T), so we may assume without loss of generality that there are at least p left-diffuse colors. We let $\Delta_L \subseteq [q]$ be a set of exactly p left-diffuse colors.

The proof in this case relies repeatedly on the following simple, but remarkably useful, observation, which follows immediately from the definition of the set X_i.

Claim 4.3. *Let i be a short color, and let $v \in X_i$. There is a set $E_i(v) \subseteq A \cup B$ with $|E_i(v)| < 2s$ such that the following holds for all $w \in (A \cup B) \setminus E_i(v)$. Vertex v is adjacent to w in G, and if $v \to w$ in T, then the color of the edge vw is i.*

Here, we think of the set $E_i(v)$ as a set of "exceptional" vertices: apart from this small set of vertices, every out-directed edge of v to $A \cup B$ receives color i.

Proof. Let $\overline{N}(v)$ be the non-neighborhood of v in G. We fix an i-avoiding directed path P in $T[A \cup B]$ ending at v of length $\ell_i(^{\to}v)$. As the color i is short, we must have $|V(P)| = \ell_i(^{\to}v) < \gamma N$. Now let $E_i(v) = V(P) \cup \overline{N}(v) \cup X_i$; we claim that this definition satisfies the desired property.

First, we have that $|E_i(v)| \leqslant |V(P)| + |\overline{N}(v)| + |X_i| \leqslant \gamma N + 4\delta N + s < 2s$. Additionally, as $\overline{N}(v) \subseteq E_i(v)$, we certainly have that v is adjacent in G to all vertices in $(A \cup B) \setminus E_i(v)$. For the final property, fix some $w \in (A \cup B) \setminus E_i(v)$ with $v \to w$. As $w \notin X_i$ (since $X_i \subseteq E_i(v)$), we have that $\ell_i(^{\to}v) \geqslant \ell_i(^{\to}w)$, for X_i consists of the s vertices with the largest value of $\ell_i(^{\to}\bullet)$. Moreover, as $w \notin E_i(v)$, we see that $w \notin V(P)$. Consider the path $P + w$, obtained from P by adding the edge $v \to w$ as the final edge. This is indeed a directed path, since $v \to w$ and $w \notin V(P)$. Additionally, if the edge vw does not receive color i, then $P + w$ is an i-avoiding directed path in $T[A \cup B]$ ending at w. Therefore its length is at most $\ell_i(^{\to}w)$, from which we find

$$\ell_i(^{\to}v) \geqslant \ell_i(^{\to}w) \geqslant |V(P + w)| = |V(P)| + 1 = \ell_i(^{\to}v) + 1,$$

a contradiction. Thus, the edge vw must receive color i. □

For $i \in \Delta_L$, let $U_i = X_i \cap A$. As the color i is left-diffuse, we know that $|X_i \cap B| < s/2$, hence $|U_i| \geqslant s/2$. The following immediate consequence of Claim 4.3 shows that the sets $\{U_i : i \in \Delta_L\}$ are pairwise disjoint.

Claim 4.4. *Let $i, j \in \Delta_L$ be left-diffuse colors with $i \neq j$. Then the sets U_i and U_j are disjoint.*

Proof. Suppose for contradiction that there is some vertex $v \in U_i \cap U_j$. Recall that $|B| = 4s$, and note that

$$|E_i(v)| + |E_j(v)| < 2s + 2s = 4s = |B|,$$

by Claim 4.3. This implies that there is some vertex $w \in B \setminus (E_i(v) \cup E_j(v))$. By Claim 4.3, we have $vw \in E(G)$. Moreover, since $v \in U_i \subseteq A$ and $w \in B$, we know that $v \prec w$, hence must have that $v \to w$ in T. Thus, by Claim 4.3, the edge vw receives color i. But for the exact same reason, it must receive color j, a contradiction. □

Let $U = \bigcup_{i \in \Delta_L} U_i$. By definition, for every $v \in U$, we must have that $v \in U_i$ for some $i \in \Delta_L$; moreover, this i is unique by Claim 4.4. Thus, for a vertex $v \in U$, we denote by $\iota(v)$ the unique index $i \in \Delta_L$ for which $v \in U_i$.

Thanks to Claim 4.3, we know a great deal about the colors of edges within U: almost all such edges are colored according to the index ι of their left endpoint. In particular, almost all edges in U are colored with colors from Δ_L. In most of the remainder of the proof, we will use this structure to show that for all colors $k \notin \Delta_L$, we can find a long k-avoiding directed path within U; this is plausible, since we know that the edges within U that *are* colored k must be quite restricted.

Our first goal in this direction is to construct an appropriate structure within U that we will use to glue long k-avoiding paths together. The precise structure we use is given in the following claim. For a vertex v and vertex subset S, we write $v \prec S$ if $v \prec w$ for all $w \in S$, and we write $S \prec v$ if $w \prec v$ for all $w \in S$.

Claim 4.5. *There exist vertices $v_1, \dots, v_{t+1} \in U$, as well as sets $S_1, \dots, S_t \subseteq U$, with the following properties.*

- *(i) We have $t = p/24$ and $|S_a| \geqslant s$ for all $a \in [t]$.*
- *(ii) We have $v_1 \prec S_1 \prec v_2 \prec S_2 \prec v_3 \prec \cdots \prec v_t \prec S_t \prec v_{t+1}$.*

(iii) S_a *is contained in the out-neighborhood of* v_a *and in the in-neighborhood of* v_{a+1}*, for all* $a \in [t]$*. That is, the edges are directed as* $v_a \to S_a \to v_{a+1}$*.*

(iv) All edges from v_a *to* S_a *and all edges from* S_a *to* v_{a+1} *are colored with a color from* Δ_L*.*

Proof. We begin by setting v_1 to be the first vertex in U according to $\prec$. We now recursively define S_a and v_{a+1}, given the value of v_a, such that the desired properties hold. Having already defined v_1, we can start the recursion.

So suppose the value of v_a is given. If there are fewer than $12s$ vertices in U which come after v_a under $\prec$, then we stop the recursion. Otherwise, we let I_a denote the next $4s$ vertices in U after v_a, and let J_a be the subsequent $8s$ vertices in U after I_a. We finally define $I'_a = I_a \setminus E_{\iota(v_a)}(v_a)$. Note that

$$4s = |I_a| \geqslant |I'_a| = |I_a \setminus E_{\iota(v_a)}(v_a)| \geqslant |I_a| - |E_{\iota(v_a)}(v_a)| \geqslant 4s - 2s = 2s,$$

that is, that $2s \leqslant |I'_a| \leqslant 4s$.

Next, consider an auxiliary bipartite graph Γ between I'_a and J_a, where we join $v \in I'_a$ to $w \in J_a$ if and only if $w \in E_{\iota(v)}(v)$. As $|E_{\iota(v)}(v)| \leqslant 2s$ by Claim 4.3, we see that in Γ, all vertices in I'_a have degree at most $2s$. Thus, the number of edges in Γ is at most $2s|I'_a| \leqslant 8s^2$. As a consequence, there must exist some vertex $w \in J_a$ whose degree in Γ is at most $(8s^2)/|J_a| = s$. We let v_{a+1} be such a choice of $w \in J_a$. Finally, we let $S_a \subseteq I'_a$ be the non-neighbors of v_{a+1} in Γ. By construction, we have

$$|S_a| \geqslant |I'_a| - s \geqslant 2s - s = s.$$

We have thus finished defining S_a and v_{a+1}, and it remains to verify the claimed properties. We have just shown that $|S_a| \geqslant s$, as claimed in (i). As $v_a \prec I_a \prec J_a$, we certainly have $v_a \prec S_a \prec v_{a+1}$, as claimed in (ii). As $S_a \subseteq I'_a \subseteq (A \cup B) \setminus E_{\iota(v_a)}(v_a)$, we conclude from Claim 4.3 that all edges from v_a to S_a are oriented as $v_a \to S_a$, and they all receive color $\iota(v_a)$, proving the first halves of (iii) and (iv). For the corresponding claims about edges from S_a to v_{a+1}, fix some $v \in S_a$. By the choice of v_{a+1}, we know that $v_{a+1} \notin E_{\iota(v)}(v)$, hence we again conclude that the edge vv_{a+1} is oriented as $v \to v_{a+1}$, and that it receives color $\iota(v) \in \Delta_L$, completing the proofs of (iii) and (iv).

Note that at every step of this process, we remove at most $12s$ vertices from U, hence we can continue this so long as $a < |U|/(12s)$. Also note that U is the union of the p disjoint sets U_i with $i \in \Delta_L$, and each such set has size at least $s/2$, by the definition of a left-diffuse color. This implies that $|U| \geqslant ps/2$, and hence we can continue the process at least to step t, where

$$t = \frac{|U|}{12s} \geqslant \frac{ps}{24s} = \frac{p}{24}. \qquad \square$$

For each $a \in [t]$, let T_a be the subtournament of T induced on the vertex set S_a. The key claim to complete the proof is the following, which allows us to glue together k-avoiding paths in all of the tournaments T_a, for every color $k \notin \Delta_L$.

Claim 4.6. *Let* $k \in [q] \setminus \Delta_L$ *be a color. We have that*

$$m_k(T) \geqslant \sum_{a=1}^{t} m_k(T_a).$$

Proof. Let P_a be a longest k-avoiding directed path in T_a, for all $a \in [t]$. We concatenate these paths into a longer path P defined as $v_1 P_1 v_2 P_2 \dots P_t v_{t+1}$. That is, we start at v_1, go to the first vertex of the directed path P_1, traverse P_1, go to v_2, and then continue in this fashion. By Claim 4.5, this gives us a directed path, since $v_a \to S_a \to v_{a+1}$ for all $a \in [t]$. Moreover, all edges $v_a \to S_a \to v_{a+1}$ are colored by colors in Δ_L, by Claim 4.5, and in particular are not colored by the color k. Thus, the path P avoids the color k, and its length is at least $\sum_{a=1}^{t} \ell_k(T_a)$; this proves that

$$\ell_k(T) \geqslant \sum_{a=1}^{t} \ell_k(T_a).$$

We now split into cases depending on whether k is long or short. If k is short, then $m_k(T) = \ell_k(T)$, hence

$$m_k(T) = \ell_k(T) \geqslant \sum_{a=1}^{t} \ell_k(T_a) \geqslant \sum_{a=1}^{t} m_k(T_a).$$

On the other hand, if k is long, then

$$m_k(T) = \gamma N \geqslant \sum_{a=1}^{t} \gamma |T_a| \geqslant \sum_{a=1}^{t} m_k(T_a),$$

where the first inequality uses that the sets $S_1, \ldots, S_t$ are pairwise disjoint subsets of $[N]$. In either case, we have the claimed bound. $\square$

From Claim 4.6 and the inequality of arithmetic and geometric means, we find that

$$m_k(T) \geqslant \sum_{a=1}^{t} m_k(T_a) \geqslant t \left(\prod_{a=1}^{t} m_k(T_a) \right)^{1/t}$$

for every color $k \in [q] \setminus \Delta_L$. On the other hand, for every color $k \in \Delta_L$, we trivially have

$$m_k(T) \geqslant \max\{m_k(T_1), \ldots, m_k(T_t)\} \geqslant \left(\prod_{a=1}^{t} m_k(T_a) \right)^{1/t}.$$

Combining these bounds, we find that

$$\begin{aligned}
\Pi(T) = \prod_{k=1}^{q} m_k(T) &= \prod_{k \in \Delta_L} m_k(T) \cdot \prod_{k \in [q]\setminus\Delta_L} m_k(T) \\
&\geqslant \prod_{k \in \Delta_L} \left(\prod_{a=1}^{t} m_k(T_a) \right)^{1/t} \cdot \prod_{k \in [q]\setminus\Delta_L} t \left(\prod_{a=1}^{t} m_k(T_a) \right)^{1/t} \\
&= t^{q-p} \left(\prod_{a=1}^{t} \prod_{k=1}^{q} m_k(T_a) \right)^{1/t} \\
&= t^{q-p} \left(\prod_{a=1}^{t} \Pi(T_a) \right)^{1/t} \\
&\geqslant t^{q-p} \cdot \Pi_q(s),
\end{aligned}$$

where the final step uses that each T_a is a q-edge-colored tournament on $|S_a| \geqslant s$ vertices.

By the induction hypothesis on s vertices, we know that $\Pi_q(s) \geqslant c_q s^{q - Cq/\sqrt{\log q}}$. From the definition of s in (7), we have that

$$\frac{s^{q-Cq/\sqrt{\log q}}}{N^{q-Cq/\sqrt{\log q}}} = \left(\frac{1}{q \cdot 2^{\sqrt{\log q}}} \right)^{q-Cq/\sqrt{\log q}} = q^{-q+Cq/\sqrt{\log q}} \cdot 2^{-q\sqrt{\log q}+Cq} \geqslant q^{-q} \cdot 2^{(C-1)q\sqrt{\log q}}.$$

Similarly, since $t = p/24$ from Claim 4.5, and from the definition of p from (7), we have that

$$t^{q-p} = \left(\frac{q}{2^{\sqrt{\log q}}} \right)^{q-p} = q^{q-p} \cdot 2^{-q\sqrt{\log q}+p\sqrt{\log q}} \geqslant q^{q-p} \cdot 2^{-q\sqrt{\log q}}.$$

Since q is sufficiently large, we have that $2^{\sqrt{\log q}} \geqslant 24\sqrt{\log q}$, and hence $p \leqslant q/\sqrt{\log q}$. Therefore,

$$t^{q-p} \geqslant q^{q-p} \cdot 2^{-q\sqrt{\log q}} \geqslant q^{q-q/\sqrt{\log q}} \cdot 2^{-q\sqrt{\log q}} = q^q \cdot 2^{-2q\sqrt{\log q}}.$$

Combining these bounds, we find that

$$\begin{aligned}\frac{\Pi(T)}{c_q N^{q-Cq/\sqrt{\log q}}} &\geqslant \frac{t^{q-p}\cdot \Pi_q(s)}{c_q N^{q-Cq/\sqrt{\log q}}} \geqslant t^{q-p}\cdot \frac{s^{q-Cq/\sqrt{\log q}}}{N^{q-Cq/\sqrt{\log q}}}\\ &\geqslant \left(q^q \cdot 2^{-2q\sqrt{\log q}}\right)\left(q^{-q}\cdot 2^{(C-1)q\sqrt{\log q}}\right)\\ &= 2^{(C-3)q\sqrt{\log q}}.\end{aligned}$$

In particular, as C is sufficiently large, then this exponent is at least $2\delta q$, hence we find that $\Pi(T) \geqslant 2^{\delta q}\cdot c_q \cdot N^{q-Cq/\sqrt{\log q}}$, as we wanted to prove in (8). This completes the proof of Case 2, and thus of Proposition 4.1. ☐

5. Concluding remarks

There are a number of natural questions left open by our work. First, recall that for $q \geqslant 4$, Proposition 1.3 implies that $F_{q,q-1}(n) = G_{q,q-1}(n)$. In other words, in the vector question, the added flexibility of choosing a $(q-1)$-comparable set (where the tournament of comparability may be arbitrary) does not add anything; the largest such set is actually a $(q-1)$-increasing sequence, i.e. whose tournament is transitive. In the setting of color-avoiding paths, Theorem 1.7 gives an approximate version of an analogous statement: a tournament all of whose color-avoiding paths are short must be $o(1)$-close to transitive. However, we do not know how to prove that the extremal construction is precisely transitive, and, in fact, do not expect this to be true.

Problem 5.1. *For some integers $q \geqslant 4$ and N, does there exist a non-transitive q-edge-colored N-vertex tournament whose longest color-avoiding path has strictly fewer than $f_{q,q-1}(N)$ vertices?*

Note that the answer to this problem is affirmative for $q = 3$, since already the old construction of Hamaker and Stein [24] gives a 2-comparable set of vectors in $[7]^3$ which has strictly more vectors than the longest 2-increasing sequence in $[7]^3$, yielding such a non-transitive tournament. However, Proposition 1.3 shows that no such vector construction can exist for $q \geqslant 4$, hence we do not know how to resolve Problem 5.1.

Another natural question left open by our work is to extend our results to the more general settings of r-comparable sets in $[n]^q$, and of directed paths in q-edge-colored tournaments which receive at most r colors, for more general parameters $1 \leqslant r < q$. For example, Theorem 1.7 allows us to find long paths that receive only 3 colors whenever we are working with a tournament that is far from transitive; however, most of our other arguments are quite specialized to the color-avoiding setting of $r = q-1$, and there appear to be serious difficulties to extending them to the most general setting. On the other hand, the simple reductions recorded in Lemma 3.8 do allow us to obtain some non-trivial results in this more general setting as a consequence of Theorem 1.6. For example, we can prove the following result, which gives a good estimate when q/r is close to 1.

Proposition 5.2. *For every $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds. If $q > r$ are integers with $r \geqslant (1-\delta)q$, then $g_{q,r}(N) \geqslant c_{q,r}N^{1-\varepsilon}$ and $f_{q,r}(N) \geqslant N^{1-\varepsilon}$ for all N, where $c_{q,r} > 0$ is a constant depending only on q and r.*

Proof. By Theorem 1.6, there exists some $p_0 = p_0(\varepsilon) \geqslant 2$ such that for all $p \geqslant p_0$, we have $g_{p,p-1}(N) \geqslant c_p N^{1-\varepsilon}$. Let $\delta = 1/p_0$, and fix integers $q > r$ with $r \geqslant (1-\delta)q$. Letting $p = \lfloor \frac{q}{q-r} \rfloor$, we see that $p \geqslant \lfloor \frac{q}{\delta q} \rfloor = p_0$. Therefore, Lemma 3.8(3) implies $g_{q,r}(N) \geqslant g_{p,p-1}(N) \geqslant c_p N^{1-\varepsilon}$, which is the claimed result upon setting $c_{q,r} = c_p$(noting that p depends only on q and r). Since $f_{q,r}(N) \geqslant g_{q,r}(N)$, we find that $f_{q,r}(N) \geqslant c_{q,r}N^{1-\varepsilon}$, which implies $f_{q,r}(N) \geqslant N^{1-\varepsilon}$ by Proposition 3.6. ☐

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